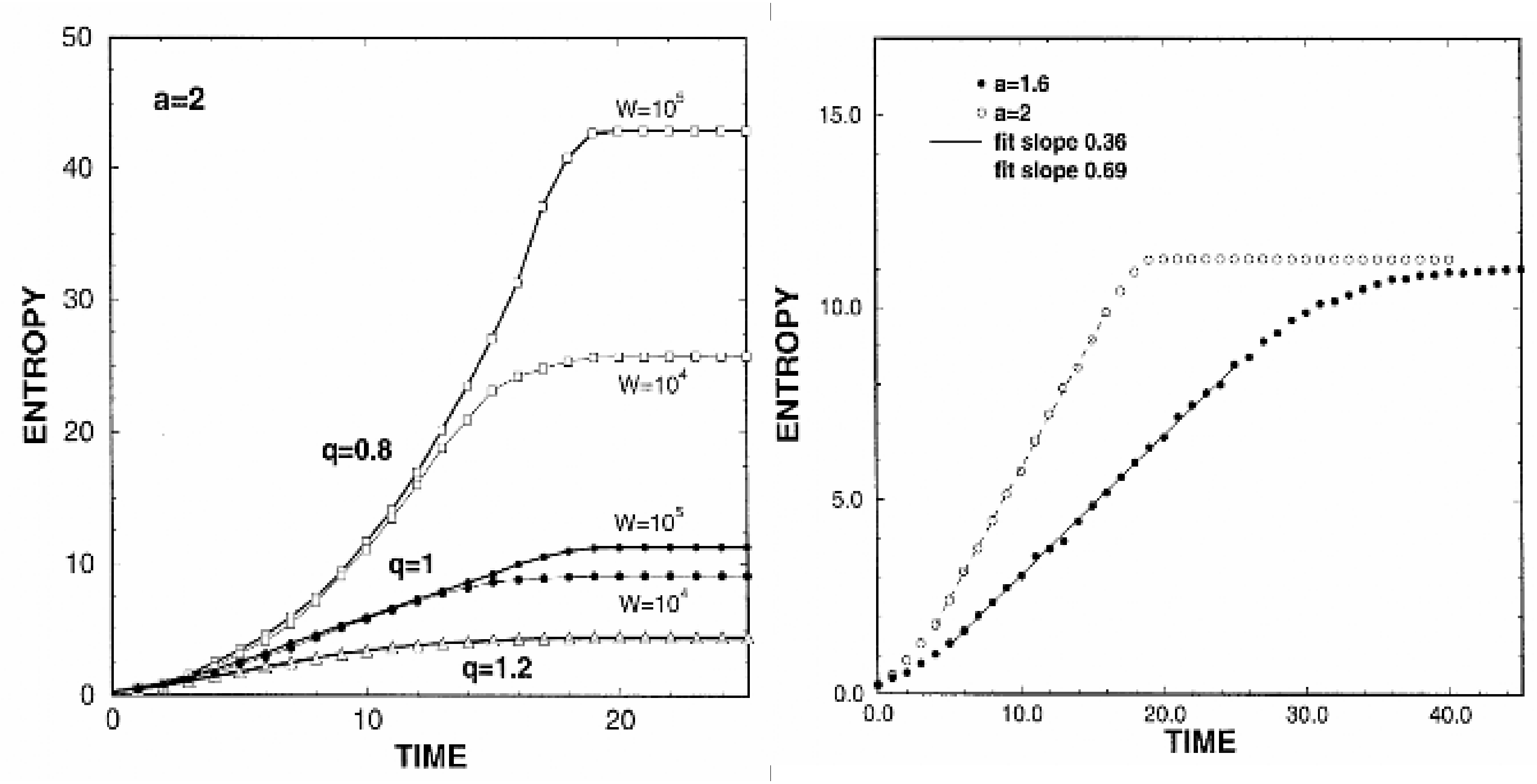
LOGISTIC MAP:

$$x_{t+1} = 1 - a\, x_t^{\,2} \quad (0 \le a \le 2;\; -1 \le x_t \le 1;\; t = 0,1,2,\ldots)$$

(strong chaos, i.e., **positive** Lyapunov exponent)



V. Latora, M. Baranger, A. Rapisarda and C. T., Phys. Lett. A **273**, 97 (2000)

We verify

$$K_1 = \lambda_1 \quad (Pesin-like\ identity)$$

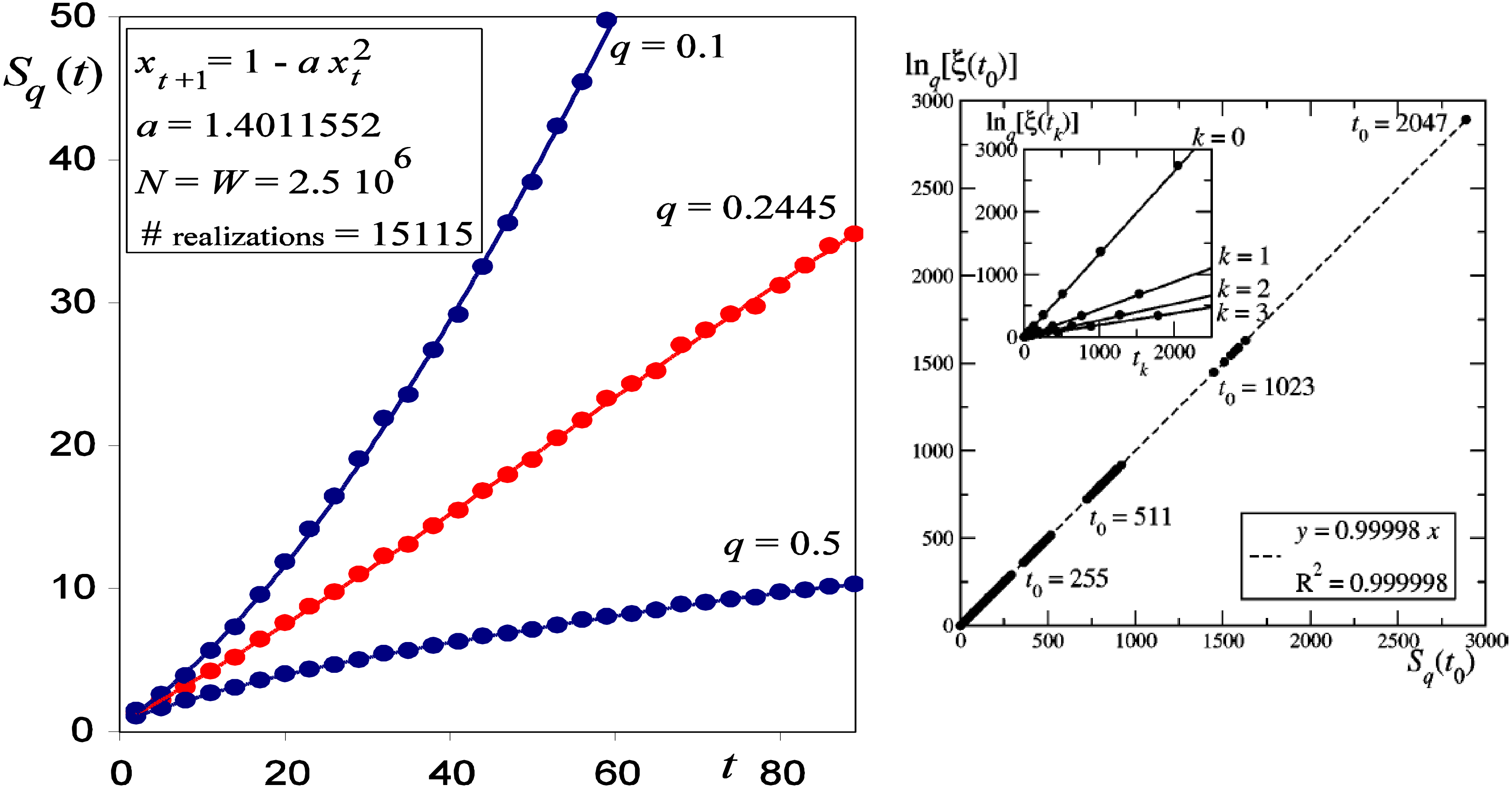
where

$$K_1 \equiv \lim_{t\to\infty} \frac{S_1(t)}{t}$$

and

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda_1 t}$$

(weak chaos, i.e., **zero** Lyapunov exponent)



C. T. , A.R. Plastino and W.-M. Zheng, Chaos, Solitons & Fractals **8**, 885 (1997)
M.L. Lyra and C. T. , Phys Rev Lett **80**, 53 (1998)
V. Latora, M. Baranger, A. Rapisarda and C. T. , Phys Lett A **273**, 97 (2000)
E.P. Borges, C. T. , G.F.J. Ananos and P.M.C. Oliveira, Phys Rev Lett **89**, 254103 (2002)
F. Baldovin and A. Robledo, Phys Rev E **66**, R045104 (2002) and **69**, R045202 (2004)
G.F.J. Ananos and C. T. , Phys Rev Lett **93**, 020601 (2004)
E. Mayoral and A. Robledo, Phys Rev E **72**, 026209 (2005), and references therein

It can be proved that

$$K_q = \lambda_q \quad (q-generalized \;\; Pesin-like\ identity)$$

where

$$K_q \equiv \lim_{t\to\infty} \sup\left\{\frac{S_q(t)}{t}\right\}$$

and

$$\xi(t) \equiv \sup\left\{\lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)}\right\} = e_q^{\lambda_q\, t}$$

with

$$\frac{1}{1-q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}} = \frac{\ln \alpha_F}{\ln 2} \quad and \quad \lambda_q = \frac{1}{1-q}$$

$$\left[x_{t+1} = 1 - a\,|\,x_t\,|^z \quad \Rightarrow \quad \frac{1}{1-q(z)} = \frac{1}{\alpha_{\min}(z)} - \frac{1}{\alpha_{\max}(z)} = (z-1)\frac{\ln \alpha_F(z)}{\ln 2} \right]$$

M.L. Lyra and C. Tsallis, Phys Rev Lett **80**, 53 (1998)

EDGE OF CHAOS OF THE LOGISTIC MAP:

(Using result in http://pi.lacim.uqam.ca/piDATA/feigenbaum.txt)

$$q = 1 - \frac{\ln 2}{\ln \alpha_F} =$$

0.244487701341282066198770423404680405234446935490057673670365098632774967276655866575515622685754070628834964038272830606360019373033181896455134108127780979219438602708319449005246581352150317453495207494044816546094908744833405672362246648808333307214231898714587299268154849677460786482183456906337020594682046189902167532145754611743830500849686040884696949170436747899150601664649106021783482788999381838252255458233803811311803180544823675794499039707439546614634081555316878853503011382149141126624632894013037015235493657147126991792102162268883302967540578063070682236881043201579035212374073544460297000605525042314202808919357881123973197797484423515245604092644670957957030465861412956647966668774368324049202275739300475089531185517955872048399269689682755585244502443652682560942378012803309487795440354252485904337976180271183000457358555073894113675878440062913563042167454169409213569860320785908819985935900731933680106996749670790445609241863211205413054739398579554441034761222259213684621934600936...

(1018 meaningful digits)

14 August 2000

PHYSICS LETTERS A

Physics Letters A 273 (2000) 97–103

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The rate of entropy increase at the edge of chaos

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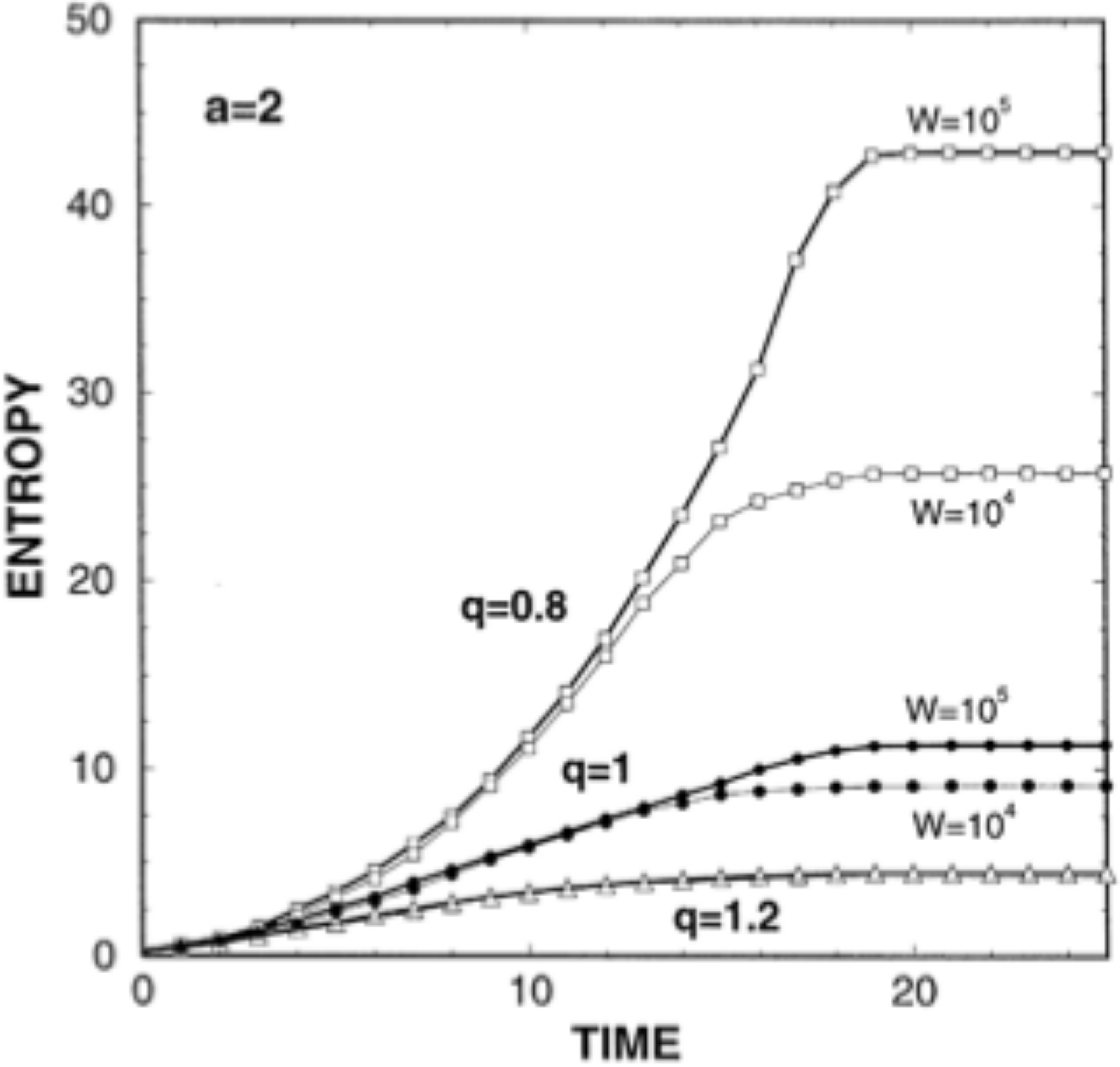
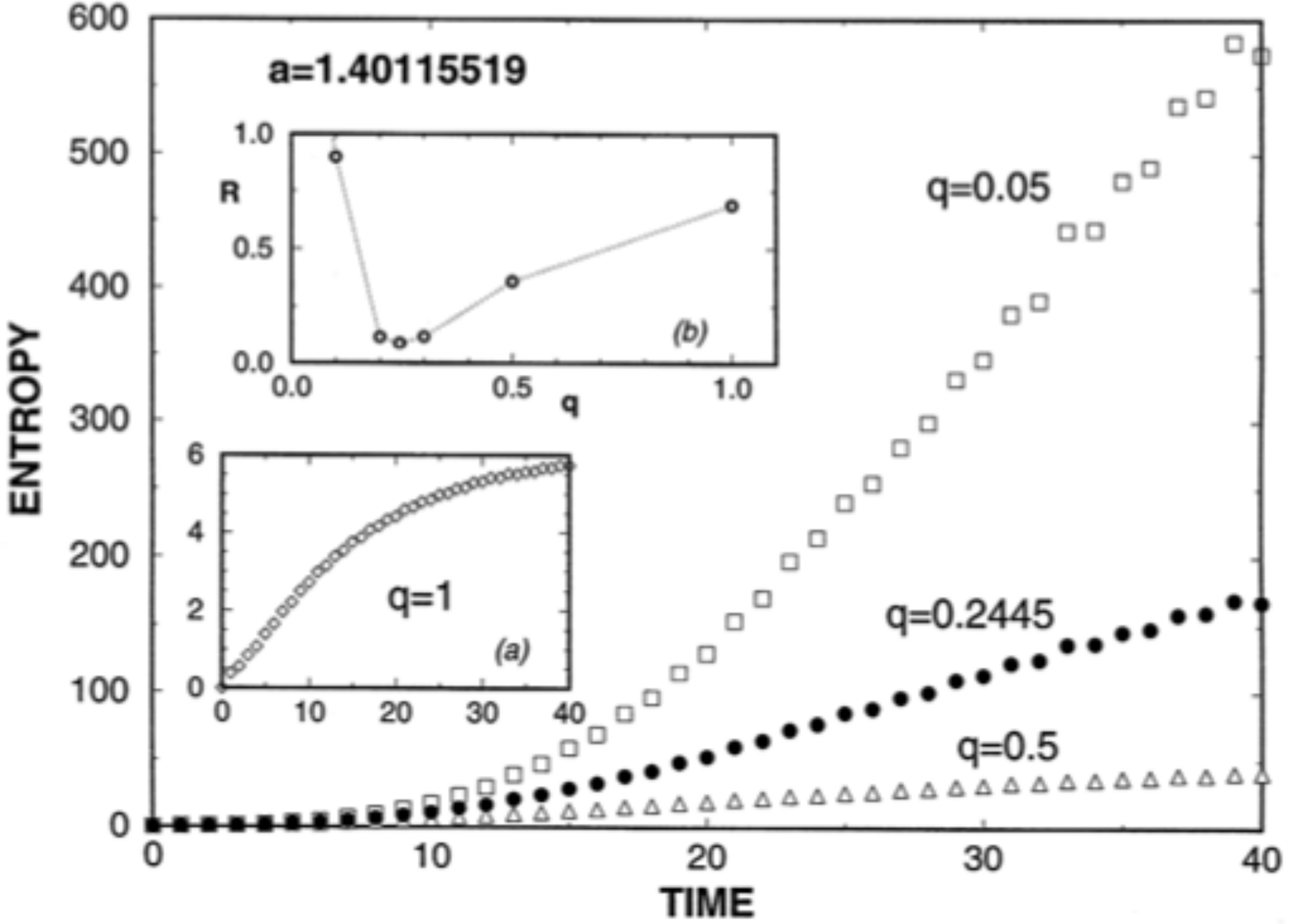
Received 27 June 2000; accepted 6 July 2000
Communicated by C.R. Doering

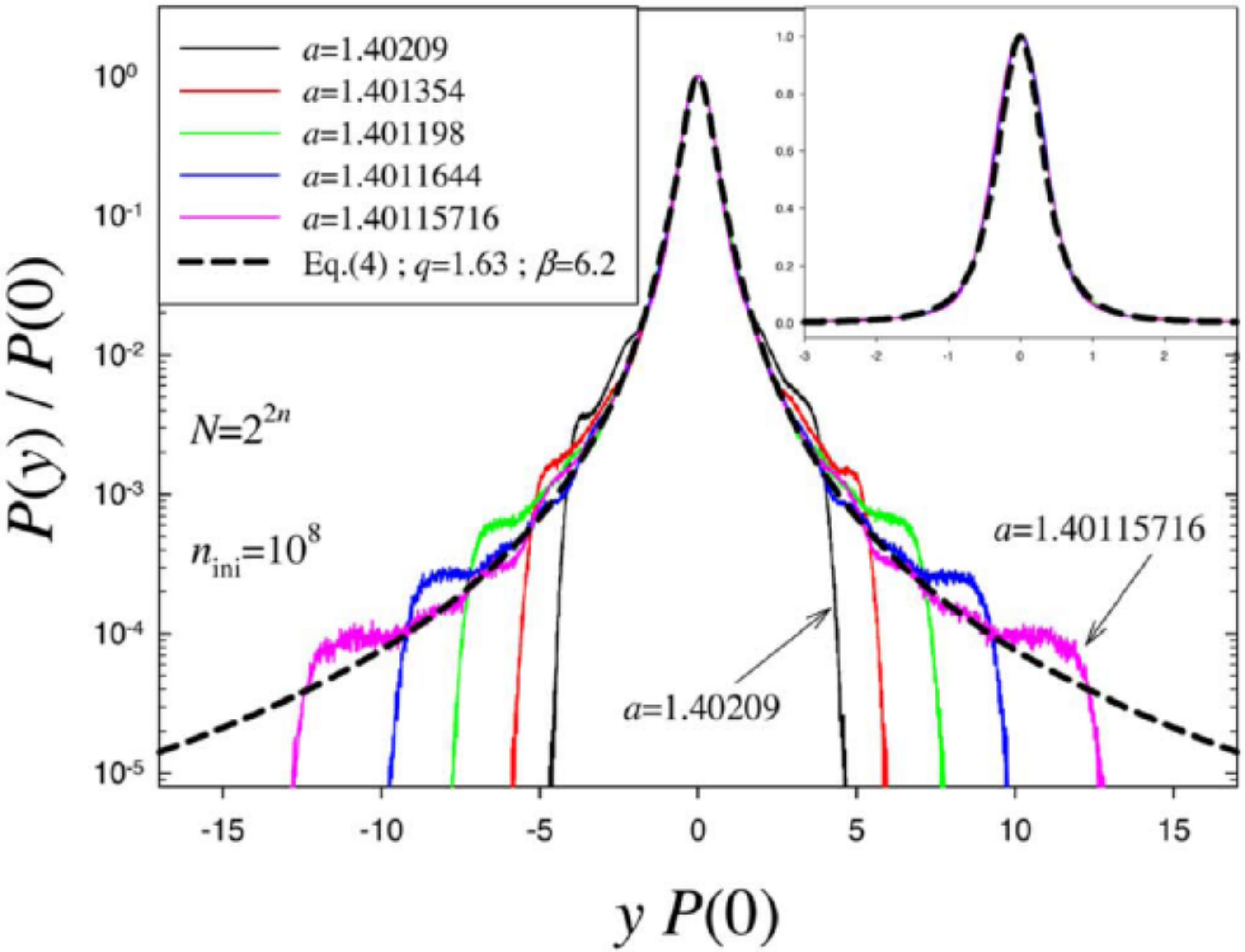


a=2
ENTROPY
TIME
W=10^5
W=10^4
q=0.8
W=10^5
q=1
W=10^4
q=1.2
0
10
20
30
40
50
0
10
20



a=1.40115519
ENTROPY
TIME
q=0.05
q=0.2445
q=0.5
q=1
(a)
(b)
R
q
600
500
400
300
200
100
0
10
20
30
40
1.0
0.5
0.0
6
4
2

LOGISTIC MAP AT THE EDGE OF CHAOS:



$q = 1.63$

$\beta = 6.2$

U. Tirnakli, C. T. and C. Beck, Phys Rev E **79** (2009) 056209

EDGE OF CHAOS OF THE LOGISTIC MAP:

$$q\text{ - triplet}\begin{cases} q_{sensitivity} = q_{entropy} = 0.2444877013412820661 98... \\ q_{relaxation} = 2.249784109... \\ q_{stationary\ state} = 1.65 \pm 0.05 \end{cases}$$

hence $q_{sens} < 1 < q_{stat} < q_{rel}$

CONJECTURE: $[\text{N.O. Baella (2010)}]$ $\varepsilon \equiv 1 - q$

$$\varepsilon_{relaxation} + \varepsilon_{sensitivity} = \varepsilon_{sensitivity}\ \varepsilon_{stationary\ state}$$

hence

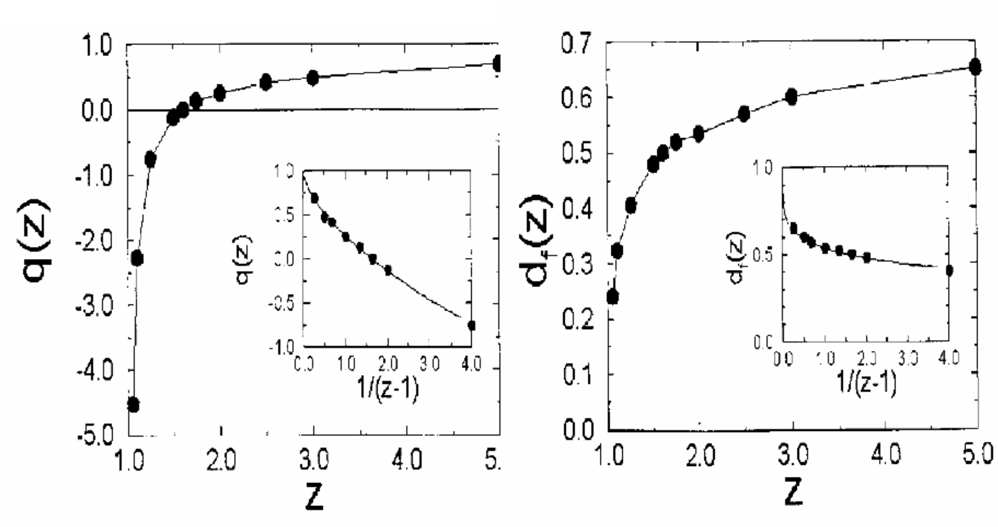
$$q_{stationary\ state} = \frac{q_{relaxation} - 1}{1 - q_{sensitivity}} = 1.65424...$$

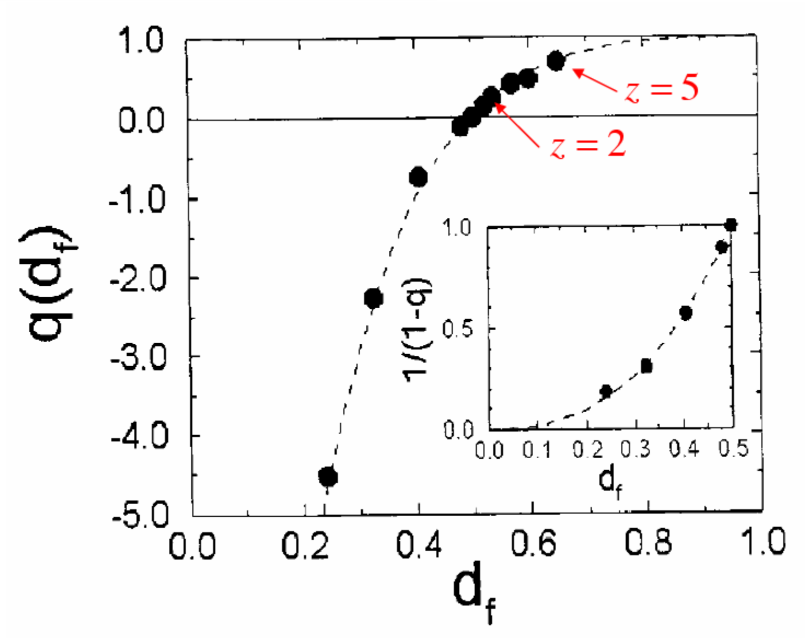
z-LOGISTIC MAP:

[U.M.S. Costa, M.L. Lyra, A.R. Plastino and C.T., Phys. Rev E 56, 245 (1997)]

$$x_{t+1} = 1 - a\,|\,x_t\,|^z$$

$x_{t+1} = 1 - a\,|\,x_t\,|^z \quad (z \geq 1;\; 0 \leq a \leq 2;\; t = 0,\; 1,\; 2,\; \ldots;\; -1 \leq x_t \leq 1)$





Nonequilibrium Probabilistic Dynamics of the Logistic Map at the Edge of Chaos

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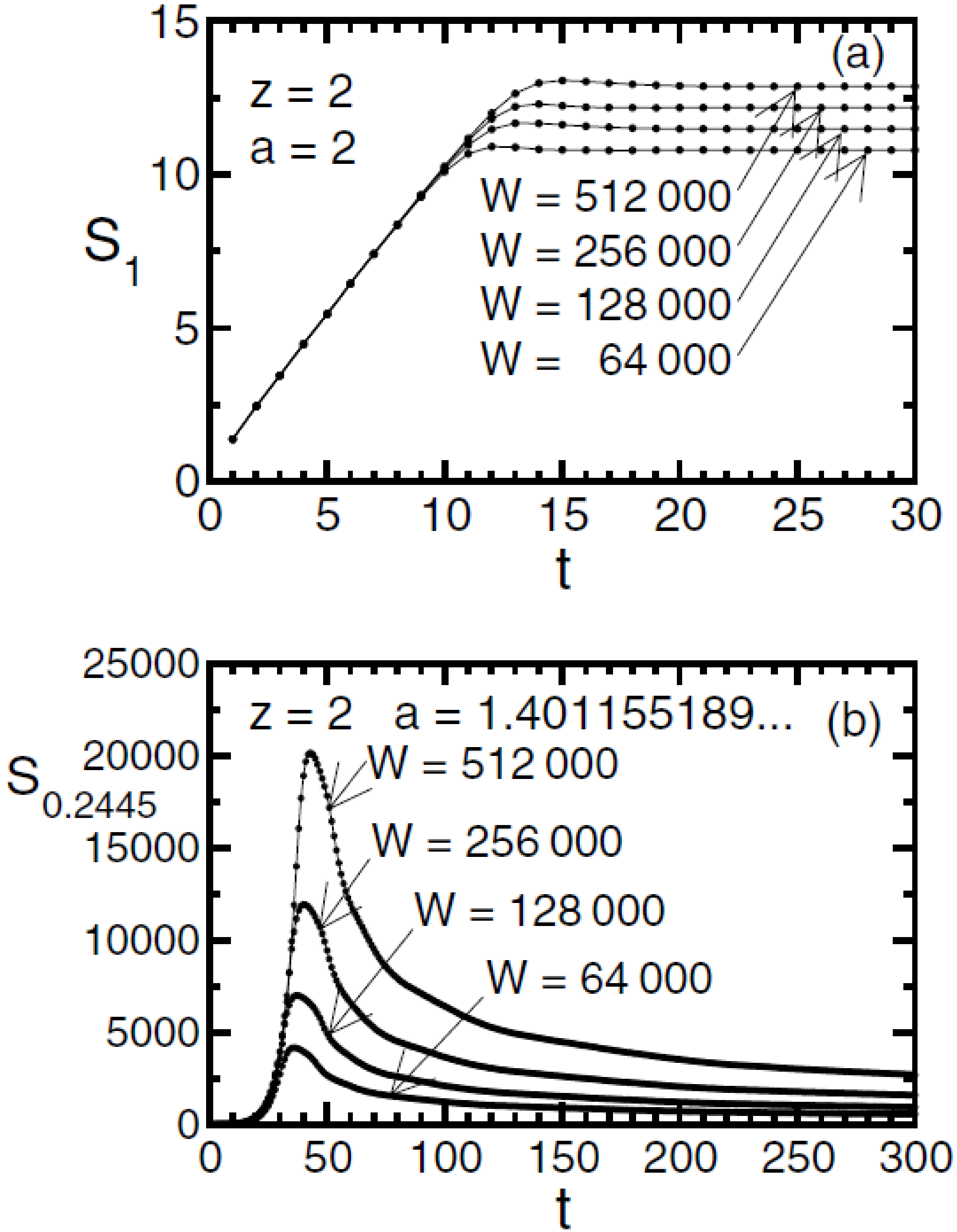
[3]*Departamento de Física, Universidad Nacional de Trujillo, Avenida Juan Pablo II, s/n, Trujillo, Peru*

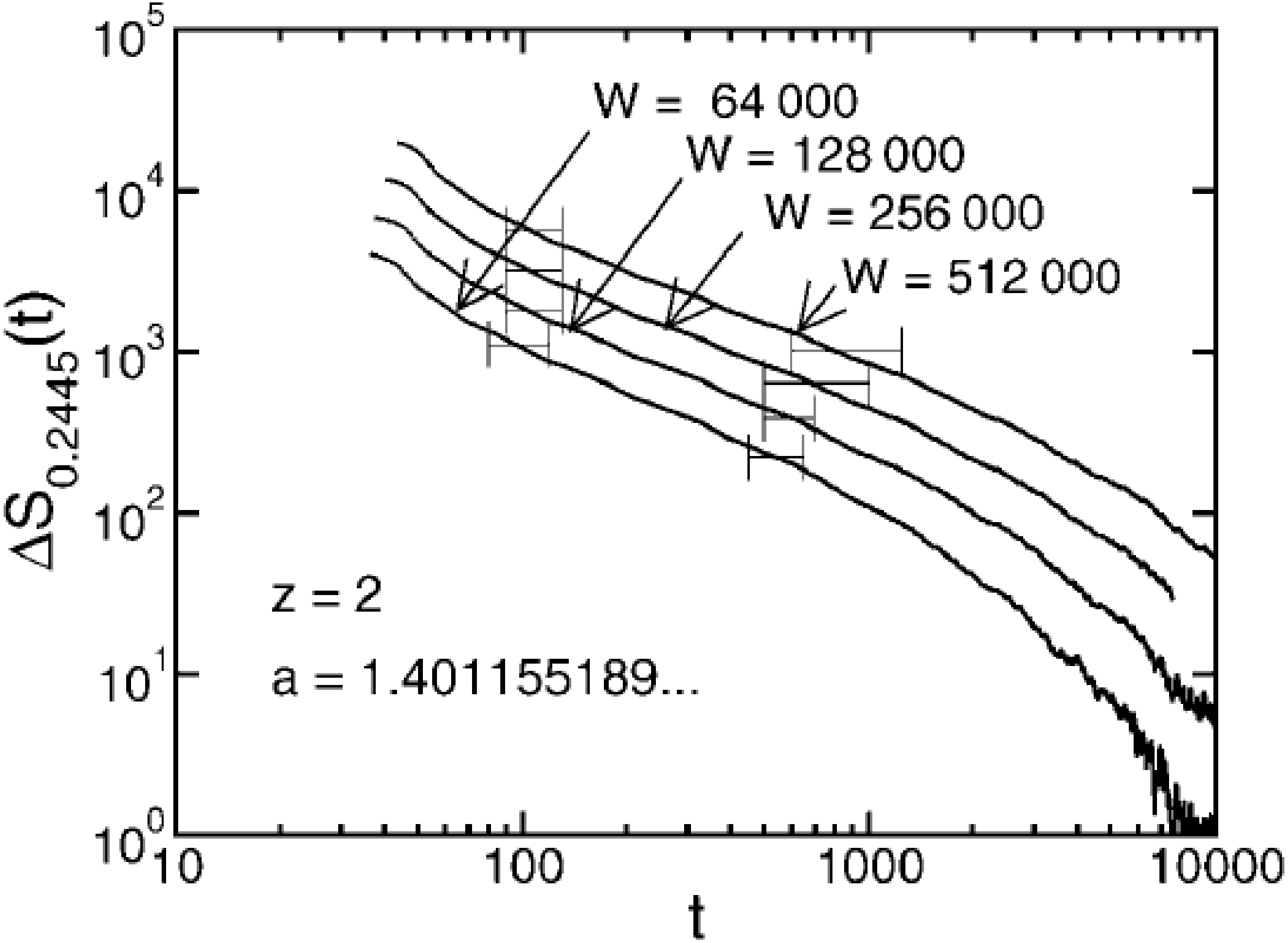
[4]*Instituto de Física, Universidade Federal Fluminense, Avenida Litorânea s/n, Boa Viagem, 24210-340, Niterói, RJ, Brazil*

(Received 16 March 2002; published 5 December 2002)

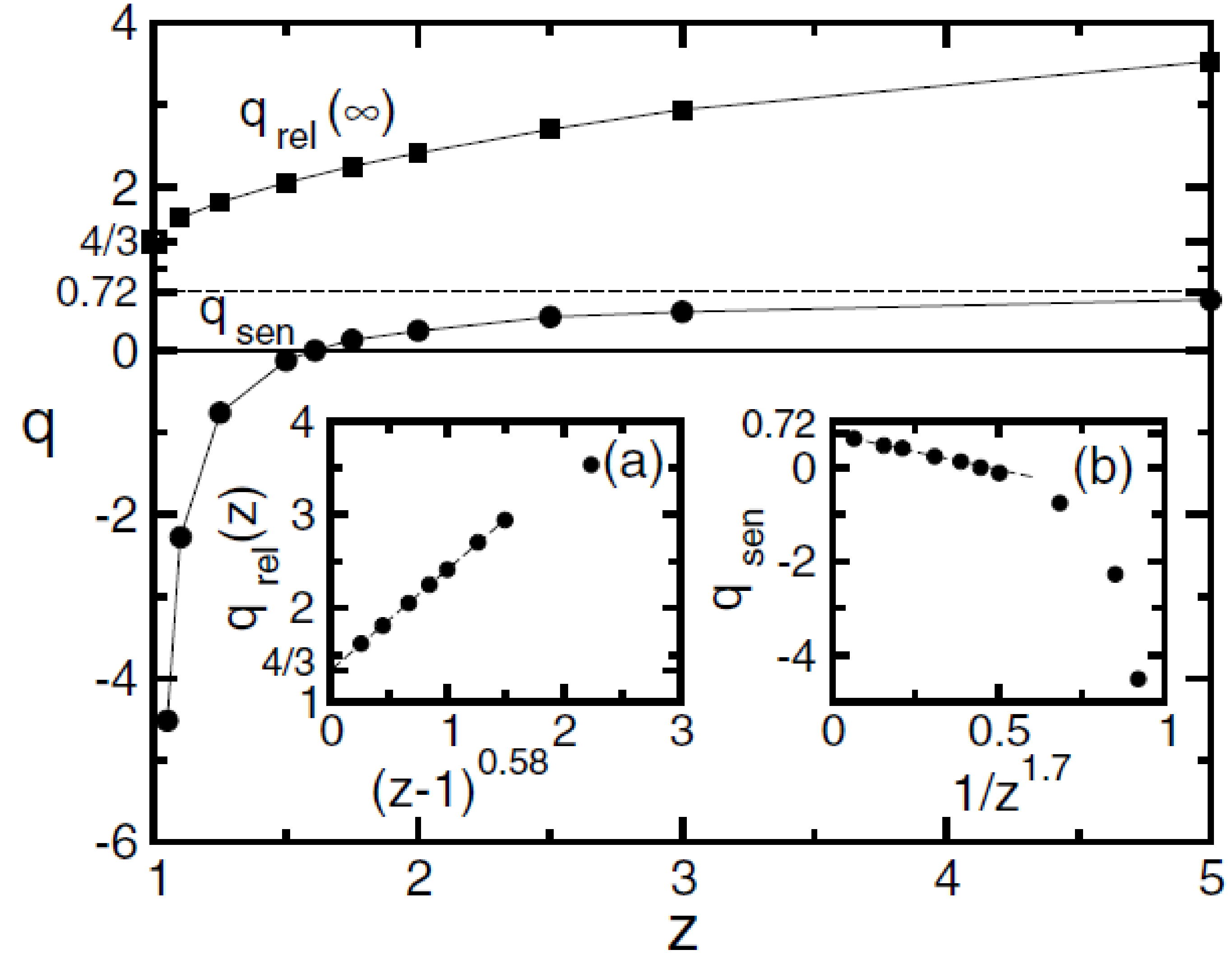
We consider nonequilibrium probabilistic dynamics in logisticlike maps $x_{t+1} = 1 - a|x_t|^z$, $(z > 1)$ at their chaos threshold: We first introduce many initial conditions within one among $W \gg 1$ intervals partitioning the phase space and focus on the unique value $q_{\rm sen} < 1$ for which the entropic form $S_q \equiv (1 - \sum_{i=1}^{W} p_i^q)/(q-1)$ *linearly* increases with time. We then verify that $S_{q_{\rm sen}}(t) - S_{q_{\rm sen}}(\infty)$ vanishes like $t^{-1/[q_{\rm rel}(W)-1]}$ $[q_{\rm rel}(W) > 1]$. We finally exhibit a new finite-size scaling, $q_{\rm rel}(\infty) - q_{\rm rel}(W) \propto W^{-|q_{\rm sen}|}$. This establishes quantitatively, for the first time, a long pursued relation between sensitivity to the initial conditions and relaxation, concepts which play central roles in nonextensive statistical mechanics.



E.P. Borges, C. T., G.F.J. Ananos and P.M.C. Oliveira
Phys Rev Lett **89**, 254103 (2002)



E.P. Borges, C. T., G.F.J. Ananos and P.M.C. Oliveira
Phys Rev Lett **89**, 254103 (2002)



z dependences of q_{sen} [from [11], or self-consistently through Eq. (3)] and $q_{\text{rel}}(\infty)$. Inset (a): $z \to 1$ extrapolation for $q_{\text{rel}}(\infty)$; $q_{\text{rel}}(\infty) = 4/3 + 1.077(z-1)^{0.58}$ fits the $z \leq 3.0$ data. Inset (b): $z \to \infty$ extrapolation for $q_{\text{sen}}(\infty)$; $q_{\text{sen}}(z) = 0.72 - 1.525/z^{1.7}$ fits the $z \geq 1.75$ data.

E.P. Borges, C. T., G.F.J. Ananos and P.M.C. Oliveira
Phys Rev Lett **89**, 254103 (2002)

VOLUME 93, NUMBER 2 PHYSICAL REVIEW LETTERS week ending 9 JULY 2004

Ensemble Averages and Nonextensivity at the Edge of Chaos of One-Dimensional Maps

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(Received 14 January 2004; published 7 July 2004)

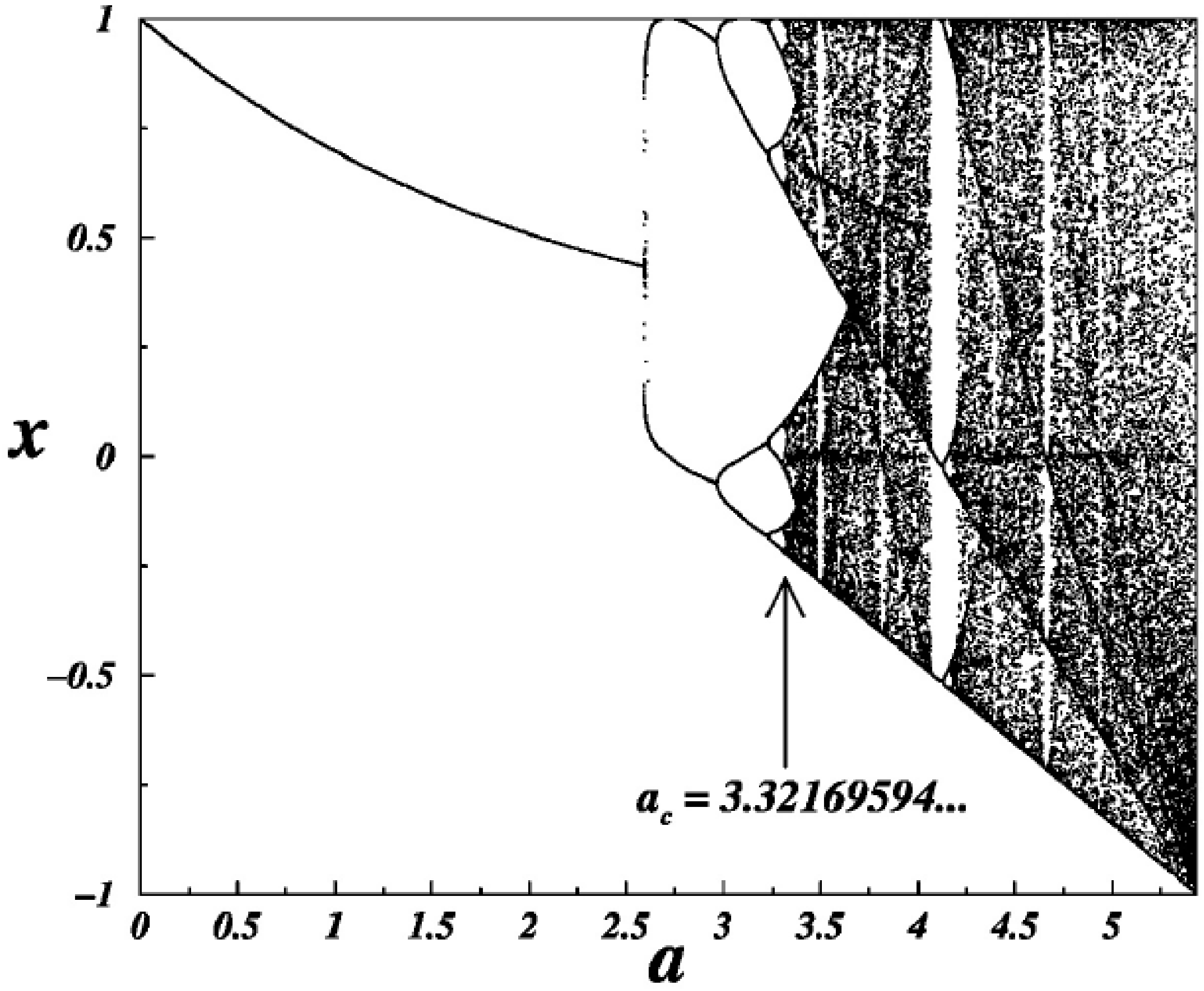
Ensemble averages of the sensitivity to initial conditions $\xi(t)$ and the entropy production per unit of time of a *new* family of one-dimensional dissipative maps, $x_{t+1} = 1 - ae^{-1/|x_t|^z}(z > 0)$, and of the known logisticlike maps, $x_{t+1} = 1 - a|x_t|^z(z > 1)$, are numerically studied, both for *strong* (Lyapunov exponent $\lambda_1 > 0$) and *weak* (chaos threshold, i.e., $\lambda_1 = 0$) chaotic cases. In all cases we verify the following: (i) both $\langle \ln_q \xi \rangle$ $[\ln_q x \equiv (x^{1-q} - 1)/(1 - q);\ \ln_1 x = \ln x]$ and $\langle S_q \rangle$ $[S_q \equiv (1 - \sum_i p_i^q)/(q - 1);\ S_1 = -\sum_i p_i \ln p_i]$ linearly increase with time for (and only for) a special value of q, $q_{\rm sen}^{\rm av}$, and (ii) the *slope* of $\langle \ln_q \xi \rangle$ and that of $\langle S_q \rangle$ coincide, thus interestingly extending the well known Pesin theorem. For strong chaos, $q_{\rm sen}^{\rm av} = 1$, whereas at the edge of chaos $q_{\rm sen}^{\rm av}(z) < 1$.

ANOTHER ONE-DIMENSIONAL DISSIPATIVE MAP:

$$x_{t+1} = 1 - ae^{-1/|x_t|^z} \qquad (z > 0; a \in [0, a^*(z)]; |x_t| \leq 1),$$

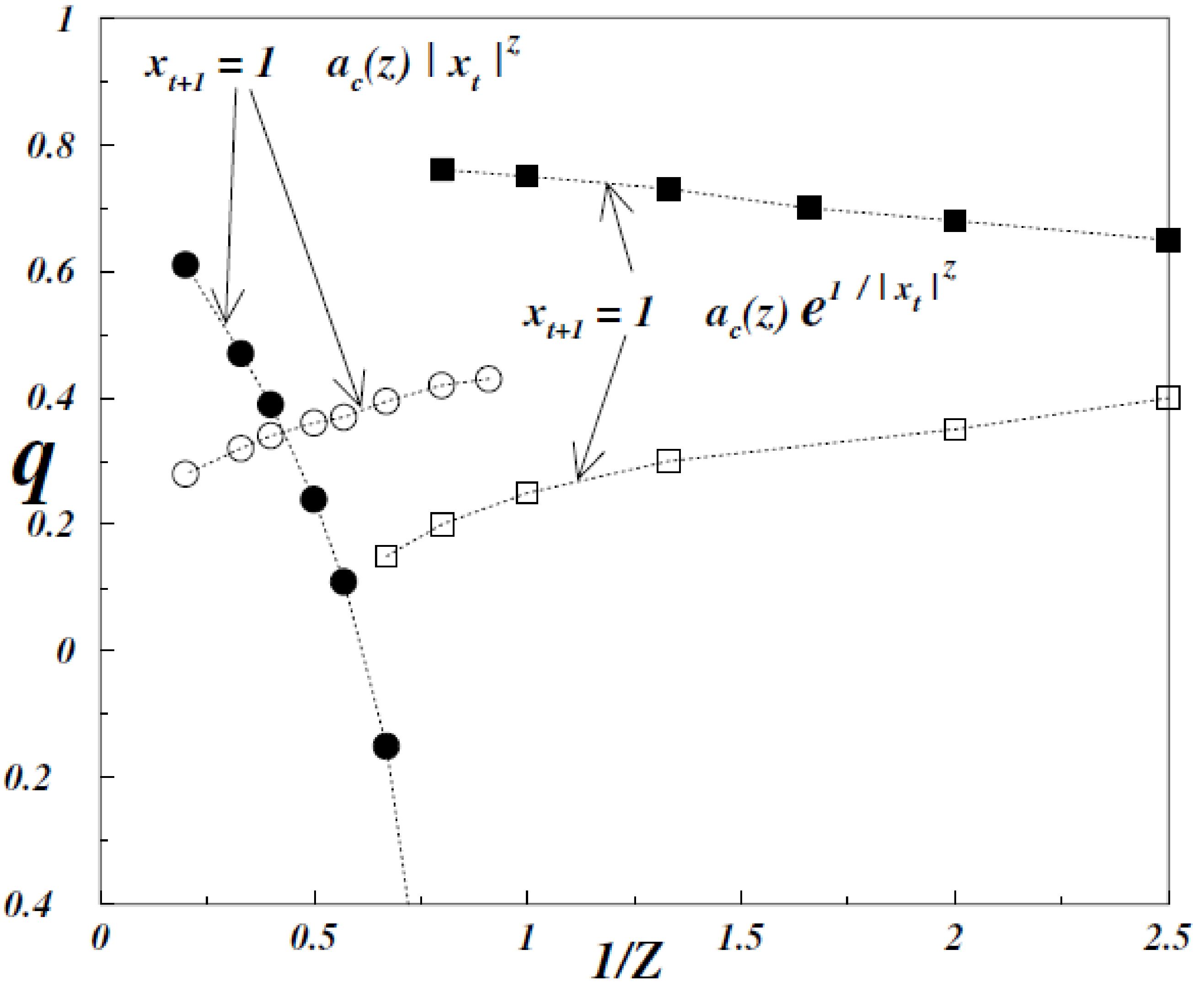
where $a^*(z)$ depends slowly from z [e.g., $a^*(0.5) \simeq 5.43$]. We address here only $z \geq z_c \simeq 0.4$, z_c being the value above which the attractors are topologically isomorphic (see Fig. 1) to those of the logistic map. In fact, one

G.F.J. Ananos and C. T., Phys Rev Lett **93**, 020601 (2004)



a dependence of the dynamical attractor of the $z = 0.5$ exponential map.

G.F.J. Ananos and C. T., Phys Rev Lett **93**, 020601 (2004)



z dependence of $q_{\mathrm{sen}}^{\mathrm{av}}$ (empty circles and squares: present work) and q_{sen} (filled circles: from [11,12]; filled squares: from [17]). Dotted lines are guides to the eye.

G.F.J. Ananos and C. T., Phys Rev Lett **93**, 020601 (2004)

BING BANG NUCLEOSYNTHESIS

The Astrophysical Journal, 834:165 (5pp), 2017 January 10
doi:10.3847/1538-4357/834/2/165

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NON-EXTENSIVE STATISTICS TO THE COSMOLOGICAL LITHIUM PROBLEM

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[5] Center for Nuclear Study, The University of Tokyo, RIKEN campus, Wako, Saitama 351-0198, Japan
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[8] National Astronomical Observatory of Japan 2-21-1 Osawa, Mitaka, Tokyo, 181-8588, Japan
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Received 2016 October 24; accepted 2016 November 25; published 2017 January 11

The Predicted Abundances for the BBN Primordial Light Elements[a]

Nuclide	Coc et al. (2012) ($q = 1$)	Cyburt et al. (2016) ($q = 1$)	Bertulani et al. (2013) ($q = 1$)	This work $q = 1$	This work $q = 1.069 \sim 1.082$	Observation
^{4}He	0.2476	0.2470	0.249	0.247	0.2469	0.2561 ± 0.0108 (Aver et al. 2010)
D/H($\times 10^{-5}$)	2.59	2.58	2.62	2.57	$3.14 \sim 3.25$	3.02 ± 0.23 (Olive et al. 2012)
^{3}He/H($\times 10^{-5}$)	1.04	1.00	0.98	1.04	$1.46 \sim 1.50$	1.1 ± 0.2 (Bania et al. 2002)
^{7}Li/H($\times 10^{-10}$)	5.24	4.65	4.39	5.23	$1.62 \sim 1.90$	1.58 ± 0.31 (Sbordone et al. 2010)

LITHIUM

Theory with q = 1 → 5.23
Theory with 1.069 < q <1.082 → 1.62 - 1.90
Observation 1.58 +- 0.31

Nonlinear Relativistic and Quantum Equations with a Common Type of Solution

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(Received 25 October 2010; published 4 April 2011)

Generalizations of the three main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations, are proposed. Nonlinear terms, characterized by exponents depending on an index q, are considered in such a way that the standard linear equations are recovered in the limit $q \to 1$. Interestingly, these equations present a common, solitonlike, traveling solution, which is written in terms of the q-exponential function that naturally emerges within nonextensive statistical mechanics. In all cases, the well-known Einstein energy-momentum relation is preserved for arbitrary values of q.

See also:

R.N. Costa Filho, M.P. Almeida, G.A. Farias and J.S. Andrade, PRA **84**, 050102(R) (2011)
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M.A. Rego-Monteiro and F.D. Nobre, Phys Rev A **88**, 032105 (2013)
I.V. Toranzo, A.R. Plastino, J.S. Dehesa and A. Plastino, Physica A **392**, 3945 (2013)
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B.G. Costa and E.P. Borges, J Math Phys **55**, 062105 (2014)
A.R. Plastino, A.M.C. Souza, F.D. Nobre and C. T., Phys Rev A **90**, 062134 (2014)
L.G.A. Alves, H.V. Ribeiro, M.A.F. Santos, R.S. Mendes and E. K. Lenzi, Physica A **429** (2015)
A.R. Plastino and C. T. EPL **113**, 50005 (2016)
A. Plastino and M.C. Rocca, EPL **116**, 41001 (2016)

$q-$generalized Schroedinger equation

(quantum non-relativistic spinless free particle)

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{2-q} \qquad (q \in R)$$

Its exact solution is given by

$$\Phi(\vec{x},t) = \Phi_0\ e_q^{i(\vec{p}\,.\,\vec{x}-Et)/\hbar} = \Phi_0\ e_q^{i(\vec{k}\,.\,\vec{x}-\omega t)}$$

with

$$E = \frac{p^2}{2m} \quad \text{(Newtonian relation!)}$$

$$E = \hbar\omega \quad \text{(Planck relation!)}$$

$$p = \hbar k \quad \text{(de Broglie relation!)}$$

$\forall q$

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)

q-generalized Klein-Gordon equation:

(quantum relativistic spinless free particle: e.g., mesons π)

$$\nabla^2 \Phi(\vec{x},t) = \frac{1}{c^2}\frac{\partial^2 \Phi(\vec{x},t)}{\partial t^2} + q\,\frac{m^2c^2}{\hbar^2}\Phi(\vec{x},t)\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{2(q-1)} \qquad (q \in R)$$

Its exact solution is given by

$$\Phi(\vec{x},t) = \Phi_0\; e_q^{\,i(\vec{p}\,.\,\vec{x}-Et)/\hbar} = \Phi_0\; e_q^{\,i(\vec{k}\,.\,\vec{x}-\omega\, t)}$$

with

$$E^2 = p^2c^2 + m^2c^4 \qquad (\forall q) \qquad \text{(Einstein relation!)}$$

Particular case: $m = 0 \;\Rightarrow\; q$-plane waves

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)

q-generalized Dirac equation:

(quantum relativistic spin $1/2$ matter and anti-matter free particles:

e.g., electron and positron)

$$i\hbar \frac{\partial\, \Phi\left(\vec{x},t\right)}{\partial t} + i\hbar c\left(\vec{\alpha}.\vec{\nabla}\right)\Phi\left(\vec{x},t\right) = \beta m c^2 A^{(q)}\left(\vec{x},t\right)\, \Phi\left(\vec{x},t\right) \qquad (q \in R)$$

with

$$\vec{\alpha} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}; \quad \beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4 \times 4 \text{ matrices})$$

$$A^{(q)}_{i\,j}\left(\vec{x},t\right) \equiv \delta_{ij}\left[\frac{\Phi_j\left(\vec{x},t\right)}{a_j}\right]^{q-1} \qquad \left(A^{(1)}_{i\,j}\left(\vec{x},t\right) = \delta_{ij}\right) \qquad (4 \times 4 \text{ matrix})$$

where $\{a_j\}$ are complex constants.

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)

Its exact solution is given by

$$\Phi(\vec{x},t) \equiv \begin{pmatrix} \Phi_1(\vec{x},t) \\ \Phi_2(\vec{x},t) \\ \Phi_3(\vec{x},t) \\ \Phi_4(\vec{x},t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} e_q^{i(\vec{p}\,.\,\vec{x}-Et)/\hbar} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} e_q^{i(\vec{k}\,.\,\vec{x}-\omega\, t)}$$

with $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ being the same $\forall q$

hence

$$E^2 = p^2c^2 + m^2c^4 \quad (q \in R)$$ (Einstein relation!)

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)